SPECTRUM OF COMPACT OPERATORS AND FIRST SPECTRAL THEOREM

From now, T compact (not necessarily selfadjoint)
Wish to describe $\sigma(T)$ putting together spectral theory and Fredholm theory

Rem $T: X \to X$ compact and $\dim X = +\infty$
$\Rightarrow 0 \in \sigma(T)$ (Indeed T is not invertible)

Rem $\{0\}$ can be
$\to$ eigenvalue: $T: \ell^2 \to \ell^2$: $Tx = \left(0, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right)$

$\to$ not eigenvalue: $\underline{T: \ell^2 \to \ell^2}$; $Tx = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right)$

but $\sigma(T) = \left\{\frac{1}{n}\right\}_{n \geq 1} \ni \{0\}$

Thm $T \in K(X)$, $\dim X = +\infty$, then

$$\sigma(T) = \{0\} \cup \sigma_p(T)$$

proof Let $\lambda \neq 0$, $T - \lambda = \left(\frac{T}{\lambda} - \mathbb{1}\right)\lambda$, $\frac{T}{\lambda}$ compact

By Fredholm:

$$\underset{\ker(T-\lambda)}{\underset{\|}{\ker\left(\frac{T}{\lambda} - \mathbb{1}\right)}} = 0 \iff \underset{\operatorname{Im}(T-\lambda)}{\underset{\|}{\operatorname{Im}\left(\frac{T}{\lambda} - \mathbb{1}\right)}} = X$$

If $\lambda \notin \sigma_p(T)$, $\lambda \neq 0 \Rightarrow \ker(T-\lambda) = \{0\}$

$\Rightarrow \operatorname{Im}(T-\lambda) = X$ $\rightsquigarrow$ $T-\lambda$ bijective
$\rightsquigarrow$ invertible with bd inverse
$\rightsquigarrow$ $\lambda \in \rho(T)$ $\square$

What about the multiplicity?

Prop Let $\lambda \in \sigma_p(T)$, $\lambda \neq 0$. Then $\dim \ker(T-\lambda) < +\infty$.

proof T compact, $\lambda \neq 0 \Rightarrow \dim \ker\left(\frac{T}{\lambda} - \mathbb{1}\right) < +\infty$ $\square$

Moreover the only accumulation point of $\sigma(T)$ is $\{0\}$.

Lemma $A \in \mathcal{L}(X)$ and $\lambda_1, \dots, \lambda_n$ distinct eigenvalues and $x_1, \dots, x_n \neq 0$ with $Ax_i = \lambda_i x_i$ (eigenv. of diff eigenvectors) then $(x_i)_{i=1}^n$ are linearly indep.

proof for $n=2$: assume they are l.d : $\exists \alpha_1, \alpha_2$:
$\alpha_1 x_1 + \alpha_2 x_2 = 0$, w.l.o.g. $\alpha_1 \neq 0$
Apply $A - \lambda_2$:

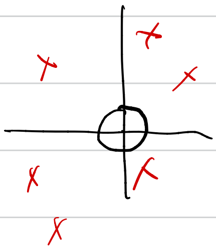
$0 = \alpha_1 (A-\lambda_2) x_1 + \alpha_2 (A - \lambda_2) x_2 = \alpha_1 (\lambda_1 - \lambda_2) x_1 \Rightarrow \alpha_1 = 0$ ↯

EXERCISE: prove it for $n \geq 3$

Prop $\forall \varepsilon > 0$, $\exists$ at most finitely many linearly independent eigenvectors corresponding to eigenvalues λ_i with $|\lambda_i| > \varepsilon$

In particular

$$\sigma_p(T) \cap \{ |\lambda| \geq \varepsilon \} \text{ is a finite set } \forall \varepsilon > 0$$



proof BC. $\exists \{x_i\}_{i=1}^{\infty}$ lin indep vectors with $T x_i = \lambda_i x_i$ and $|\lambda_i| \geq \varepsilon$ $\forall i$

Set $E_k = \operatorname{span}(x_1, \dots, x_k) \subsetneq E_{k+1}$

Riesz lemma: $\forall k$, $\exists y_k \in E_k$: $\begin{cases} \|y_k\| = 1 \\ \operatorname{dist}(y_k, E_{k-1}) \geq \frac{1}{2} \end{cases}$

Claim $\left\{ T \frac{y_k}{\lambda_k} \right\}_{k \geq 1}$ does not have any Cauchy subseq.

Then since $\left\| \frac{y_k}{\lambda_k} \right\| \leq \frac{1}{|\lambda_k|} \leq \frac{1}{\varepsilon}$ $\forall k$

this contradicts the compactness of T ↯

proof of claim: as $y_k \in E_k \rightsquigarrow y_k = \sum_{i=1} a_i x_i$

$$\rightsquigarrow T\frac{y_k}{\lambda_k} = \sum_{i=1}^{k} \frac{a_i}{\lambda_k} \lambda_i x_i = \underbrace{a_k x_k}_{\in E_k} + \underbrace{\sum_{i=1}^{k-1} a_i \frac{\lambda_i}{\lambda_k} x_i}_{\in E_{k-1}}$$

$$= y_k + z_k \,, \quad z_k \in E_{k-1}$$

So now for $k > m$

$$T\frac{y_k}{\lambda_k} - T\frac{y_m}{\lambda_m} = \underbrace{y_k}_{\in E_k} + \underbrace{z_k}_{\in E_{k-1}} \underbrace{- y_m - z_m}_{\in E_m \subseteq E_{k-1}}$$

$$\rightsquigarrow \left\| T\frac{y_k}{\lambda_k} - T\frac{y_m}{\lambda_m} \right\| = \| y_k + \underbrace{\quad\quad}_{\in E_{k-1}} \| \geq \frac{1}{2}$$

$\square$

Cor $T \in K(X)$. If $\{\lambda_k\}_{k \geq 1} \subseteq \sigma(T) \setminus \{0\}$ with λ_k distinct, then $\lambda_k \xrightarrow{k \to \infty} 0$.

($\{0\}$ is the only possible accumulation point for eigenvalues)

proof If $(\lambda_k)_k$ are distincts, the corresponding eigenvectors must be lin. indep. But there are only finitely many l.i. eigenvectors corresponding to eigenvalues with $|\lambda_k| > \varepsilon$, $\forall \varepsilon > 0$.

Compact and selfadjoint operators

Let H Hilbert, $T = T^*$, compact

what can we say about the structure of T?

Recall that in fin dim, if A is symmetric matrix then A can be diagonalized with orthonormal basis of eigenvectors

$$\begin{cases} Ax = \sum_{i=1}^{n} \lambda_i \langle x, e_i \rangle e_i \\ A\vec{e_i} = \lambda_i \vec{e_i} \end{cases}$$

We prove that a compact selfadjoint op is diagonalizable in an analogous way

thm (Spectral thm for compact selfadjoint ops)

$T \in \mathcal{K}(H)$, H Hilbert, $T = T^*$, $T \neq 0$, then

(1) $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$, $0 \in \sigma(T)$

(2) Eigenvalues are at most countable and
- either are finitely many
- or accumulate to 0

(3) Eigenspaces are pairwise orthogonal and fin dim. for not zero eigenvalues

(4) $\exists$ set of non zero eigenvalues

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$$

and countable ON system $(e_i)_{i\geq 1}$ of eigenvectors with $Te_i = \lambda_i e_i$ st

(a) $\forall x \in H$, $x = y + \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$, $y \in \ker T$

(b) $Tx = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i \rangle e_i$ (diagonal form)

and $\forall z \in \overline{\operatorname{Im} T}$ we have $z = \sum_i \langle z, e_i\rangle e_i$

i.e. $(e_i)_{i\geq 1}$ are a ON basis for $\operatorname{Im} T$

(5) If H separable, $\exists$ ON basis of H made of eigenvectors

Before the proof, let us state this lemma

Lemma If $T \in K(H)$, $T = T^*$, $T \neq 0 \Rightarrow \exists \lambda \in \sigma_p(T)$:

$$\|T\| = |\lambda|$$

<u>proof</u>

$T = T^* \Rightarrow \|T\|$ or $-\|T\| \in \sigma(T)$

$T \neq 0 \Rightarrow \|T\| \neq 0$

$T \in K(H) \Rightarrow \sigma(T)\setminus\{0\} = \sigma_p(T)\setminus\{0\}$

$\Rightarrow \|T\|$ or $-\|T\| \in \sigma_p(T)\setminus\{0\}$

$\square$

<u>proof of spectral theorem</u> (1) – (3) already proved

(4)

Step 1 Build a seq $\{\lambda_i\}$, $\{e_i\}$ of eigenvalues/eigenvectors with $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_i| \geq \dots$

<u>vector e_1</u>: look for e_1: $Te_1 = \lambda_1 e_1$, $\|e_1\| = 1$

Just take λ_1 fulfilling $\|T\| = |\lambda_1|$ (by Lemma)
and recall that $\lambda_1 \in \sigma_p(T) \setminus \{0\}$, hence we have
eigenvector $\vec{e}_1$ (not necessarily unique)

Let now consider $(e_i)_{i=1}^{n}$ defined by inductive procedure, namely

$$\begin{cases} T e_i = \lambda_i e_i \\ \|e_i\| = 1 \end{cases}, \quad |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$$

Let $E_n = \operatorname{span}(e_1, \dots, e_n)$

E_n is invariant subspace for T: $T : E_n \to E_n$,
so also $E_n^{\perp}$ is invariant subspace of T, and
an Hilbert space with the same scalar product of $\mathcal{H}$.
Define $T_{n+1} := T|_{E_n^{\perp}} : E_n^{\perp} \to E_n^{\perp}$

T_{n+1} is selfadjoint and compact on the
Hilbert space $E_n^{\perp}$ (exercise: prove it!)

Case 1 $\quad T_{n+1} = 0 \rightsquigarrow$ we stop
we set $\forall x \in H$, $x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i + y_n$
with $y_n \in E_n^{\perp} \Rightarrow T y_n = T|_{E_n^{\perp}} y_n = T_{n+1} y_n = 0$

Case 2 $\quad T_{n+1} \neq 0$. then we use again lemma, and
find λ_{n+1} st

$$\|T_{n+1}\| = |\lambda_{n+1}|$$

$\exists\, e_{n+1}$: $\quad T e_{n+1} = \lambda_{n+1} e_{n+1}, \quad \|e_{n+1}\| = 1$

We show that $|\lambda_n| \geq |\lambda_{n+1}|$
Indeed $E_{n-1} \subseteq E_n \rightsquigarrow E_n^{\perp} \subseteq E_{n-1}^{\perp}$, thus

$$|\lambda_{n+1}| = \|T_{n+1}\| = \sup_{\|x\| \leq 1,\, x \in E_n^{\perp}} \|T_{n+1} x\| = \sup_{\substack{\|x\| \leq 1 \\ x \in E_n^{\perp}}} \|Tx\|$$

$$\leq \sup_{\substack{\|x\| \leq 1 \\ x \in E_{n-1}^{\perp}}} \|Tx\| = \|T_n\| = |\lambda_n|$$

We iterate this process. If for some n we end in case 1 we stop.

otherwise we get ∞ seq $\{\lambda_n\}$ of eigenv. of T with $\lambda_n \to 0$

Step 2 Show that $x - \sum \langle x, e_i \rangle e_i \in \ker T$

By construction the $(e_n)_{n \geq 1}$ are orthogonal, thus $\sum_{i \geq 1} \langle x, e_i \rangle e_i$ converges in $\mathcal{H}$ to $x_\infty \in H$.

Indeed, by Bessel inequality,

$$\Big\| \sum_{i \geq 1} \langle x, e_i \rangle e_i \Big\|^2 = \sum |\langle x, e_i \rangle|^2 \leq \|x\|^2 < \infty$$

So write $x = x_\infty + x - x_\infty$

and we need to show $x - x_\infty \in \ker T$.

Let us put $y_n := x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i$

and clearly $y_n \to x - x_\infty$, and since

T is continuous: $T y_n \to T(x - x_\infty)$ as $n \to \infty$

We show that $T y_n \to 0$, then $x - x_\infty \in \ker T$.

$$\|T y_n\| \leq \|T|_{E_n^\perp} y_n\| \leq \|T|_{E_n^\perp}\| \|y_n\|$$

$$\leq |\lambda_{n+1}| \Big(\|x\| + \Big\| \sum_{i=1}^{n} \langle x, e_i \rangle e_i \Big\| \Big)$$

Bessel

$$\leq 2 |\lambda_{n+1}| \|x\| \xrightarrow{n \to \infty} 0$$

Thus shows that $Ty_n \to T(x-x_0)$, $Ty_n \to 0$ $\Rightarrow$ $x - x_0 \in \ker T$

$\rightsquigarrow$ $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i + \overset{\overset{x-x_0}{\parallel}}{y}$, $y \in \ker T$

Then, since $(e_i)_{i \geq 1}$ are eigenvectors of T:

$$Tx = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i \rangle e_i .$$

$\Rightarrow$ $\operatorname{Im} T \subseteq \overline{\operatorname{span}(e_i)}$ $\rightsquigarrow$ $\overline{\operatorname{Im} T} \subseteq \overline{\operatorname{span}(e_i)}$

but since $e_i = \frac{1}{\lambda_i} T e_i \in \operatorname{Im} T$ $\rightsquigarrow$ $\overline{\operatorname{Im} T} = \overline{\operatorname{span}(e_i)_{i\geq 1}}$

$\rightsquigarrow$ $(e_i)_{i \geq 1}$ ON basis of $\overline{\operatorname{Im} T}$

(5) We proved $H = \ker T \oplus \overline{\operatorname{Im} T}$, so only consider $\ker T \neq 0$.

H separable just take ON basis $\{f_i\}_{i \geq 1}$ of $\ker T$

$T f_i = 0$ $\rightsquigarrow$ f_i are eigenvectors

$\rightsquigarrow$ $\{e_i\}_{i\geq 1} \cup \{f_i\}_{i \geq 1}$ ON basis of H.

Applications to integral operators

Thm $H = L^2([a,b])$, $K(\cdot,\cdot) \in L^2([a,b]^2)$ with $K(s,t) = \overline{K(t,s)}$. Define

$$T \in \mathcal{K}(H), \quad T = T^*, \quad (Tf)(t) = \int_a^b K(t,s) f(s)\, ds$$

Take $\{e_i\}_{i\geq 1}$ eigenvectors with $Te_i = \lambda_i e_i$, then

$$K(t,s) = \sum \lambda_i\, e_i(t)\, \overline{e_i(s)} \quad \text{with conv. in } L^2$$

It follows: $\displaystyle\sum |\lambda_i|^2 = \iint |K(t,s)|^2\, ds\, dt < \infty$

Rem Integral op defined by kernel functions are a subclass of compact selfadjoint op: $\sum |\lambda_i|^2 < +\infty$
$\rightsquigarrow \lambda_i \to 0$ quickly. e.g. no such op with $\lambda_n = \frac{1}{\sqrt{n}}$

Def (Hilbert – Schmidt) $T \in \mathcal{L}(H)$, $T = T^*$ is HS if $\exists$ ON basis $(e_i)_{i\geq 1}$: $\displaystyle\sum_i \|Te_i\|^2 < +\infty$

Ex Integral op with symmetric kernel: $\displaystyle\sum_i \|Te_i\|^2 = \sum |\lambda_i|^2 < \infty$

proof Let $\eta_i(t,s) := e_i(t)\,\overline{e_i(s)}$ this is ON system in $L^2([a,b]^2)$ (not complete)

Then

$$\phi(t,s) := \sum_i \langle K, \eta_i \rangle_{L^2([a,b]^2)}\, \eta_i \in L^2 \quad \text{by Bessel}$$

(i.e. the series converges). Moreover

$$\langle K, \eta_i \rangle_{L^2([a,b]^2)} = \iint K(t,s)\, \overline{e_i(t)\, \overline{e_i(s)}}\, ds\, dt$$

$$= \int \left(\int K(t,s)\, e_i(s)\, ds \right) \overline{e_i(t)}\, ds = \langle Te_i, e_i \rangle = \lambda_i$$

$\Rightarrow \phi(t,s) = \sum_i \lambda_i\, e_i(t)\, \overline{e_i(s)}$. We show that

$K = \phi$, thus concluding.

Take $u, v \in L^2([a,b])$. By spectral thm

$$Tv = \sum \lambda_i \langle v, e_i \rangle e_i$$

$$\leadsto \langle Tv, u \rangle = \sum_i \lambda_i \langle \langle v, e_i \rangle e_i, u \rangle =$$

$$= \sum_i \lambda_i \langle v, e_i \rangle \langle e_i, u \rangle$$

$\iint K(t,s) v(s) \overline{u(t)}\, ds\, dt$

$$= \sum_i \lambda_i \langle e_i(t)\, \overline{e_i(s)}, u(t)\, \overline{v}(s) \rangle_{L^2([a,b]^2)}$$

$\langle K, u(t)\, \overline{v}(s) \rangle_{L^2([a,b]^2)}$

$$= \langle \phi, u \overline{v} \rangle_{L^2([a,b]^2)} \quad \forall\, u, v \in L^2$$

Now $\{ u(t)\, \overline{v}(s) : u, v \in L^2([a,b]) \}$ is complete in $L^2([a,b]^2)$

$$\Rightarrow \quad K = \phi$$ ◻

FUNCTIONAL CALCULUS FOR COMPACT SYMMETRIC OPERATORS

Given $p(t) = \sum_{k=1}^{N} c_k t^k$ polynomial, we can define

$$p(A) := \sum_{k=1}^{N} c_k A^k$$

What about a general function f? $f(A)$?

Let us start with compact symmetric operators

Take $f \in \mathcal{B}(\sigma(A)) = \{ f : \sigma(A) \to \mathbb{C} \,;\, \text{bounded} \}$

If A compact, by spectral thm $\exists$ ON basis of eigenvectors (H separable) st

$$A x = \sum \lambda_i \langle x, e_i \rangle e_i = \sum \lambda_i P_{\lambda_i} x$$

where P_{λ_i} is the $\perp$ projection on $\ker(A - \lambda_i)$

Let $x = \sum_{n \geq 1} x_n e_n$ with $\underbrace{(e_n) \text{ ON basis}}_{\text{made of eigenvectors}}$ of H. Put

$$f(A) x_i = \sum_{n \geq 1} f(\lambda_n) x_n e_n$$

Thm $A \in K(H)$, $A = A^*$. There is a map

$$\phi : C(\sigma(A)) \longrightarrow \mathcal{L}(H)$$

$$f \longmapsto \phi(f) := f(A)$$

$// \quad \{ f : f(\lambda_i) \to f(0) \,;\, \lambda_i \to 0 \}$

fulfilling

(i) Φ is algebraic $*$-homomorphism, that is

$$\phi(fg) = \phi(f)\,\phi(g)$$
$$\phi(\lambda f) = \lambda\, \phi(f) \qquad \forall \lambda \in \mathbb{C}$$
$$\phi(1) = \mathbb{I}$$
$$\phi(\bar{f}) = \phi(f)^*$$

(ii) If $f(x) = x \implies \phi(f) = A$

(iii) Isometry: $\|f(A)\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)|$

(iv) spectral mapping property: $\sigma(f(A)) = f(\sigma(A))$

(v) If $Ax = \lambda x$ (i.e. eigenvalues)

$$f(A)x = f(\lambda)x$$

(vi) $f \geq 0 \implies f(A) \geq 0$ i.e. $(f(A)x, x) \geq 0 \ \forall x$

<u>proof</u> EXERCISE!

<u>EX</u> T compact, $T = T^*$ $f \in \mathcal{B}(\sigma(T)) = \{ f : \sigma(T) \to \mathbb{C}, \text{ bounded} \}$

$\| f(A) \| = \sup_{\lambda \in \sigma_p(A)} |f(\lambda)|$, $\sigma_p(f(A)) = f(\sigma_p(A))$

Variational method to compute eigenvalues

By previous thm $Tx = \sum \lambda_i \langle x, e_i \rangle e_i$,

$$x = \sum \langle x, e_i \rangle e_i + y, \quad y \in \ker T$$

$$\implies \langle Tx, x \rangle = \sum_{i=1}^{\infty} \lambda_i |\langle x, e_i \rangle|^2$$

$$= \sum_i \lambda_i^+ |\langle x, e_i \rangle|^2 + \sum_i \lambda_i^- |\langle x, e_i \rangle|^2$$

$\lambda_1^+ \geq \lambda_2^+ \geq \dots \geq 0$ positive

$\lambda_1^- \leq \lambda_2^- \leq \dots \leq 0$ negative

<u>Cor</u> H ∞-dim Hilb space. Then

(i) If a pos eigenv. exists, $\lambda_1^+ = \max_{\|x\|=1} \langle Tx, x \rangle$

neg eigenv ", $\lambda_1^- = \min_{\|x\|=1} \langle Tx, x \rangle$

(ii) $\langle Tx, x \rangle \geq 0 \ \forall x \in H \iff \nexists$ negative eigenv.

<u>proof</u> (i) take $\|x\| = 1$ then

$\langle Tx,x\rangle \leq \sum_i \lambda_i^+ |\langle x, e_i\rangle|^2 \leq \lambda_1^+ \sum |\langle x,e_i\rangle|^2 \leq \lambda_1^+$

and $\langle T e_i^+, e_i^+\rangle = \lambda_1^+$

For neg eigenv. the same.

(ii) $\Leftarrow$) clear
$\Rightarrow$) It follows from (i)

$\square$

Cor (Courant–Fischer) $T \in \mathcal{L}(H)$, $T = T^*$, compact.
Assume $\lambda_{n+1}^+ > 0$ (i.e. $\exists$ at least $n+1$ pos. eigenv)
Then

(i) $$\lambda_{n+1}^+ = \min_{V:\ \dim V = n} \ \sup_{\|x\|=1,\ x \in V^\perp} \langle Tx,x\rangle$$

(ii) $$\lambda_{n+1}^+ = \max_{V:\ \dim V = n+1} \ \min_{x \in V,\ \|x\|=1} \langle Tx,x\rangle$$

proof (i) $(\leq)$ $V = \mathrm{span}(x_1,\dots,x_n)$, $x_j \in H$

CLAIM: $\exists\, y \in \mathrm{span}(e_1^+,\dots,e_{n+1}^+)$: $\left|\begin{array}{l}\|y\|=1\\ y \perp V\end{array}\right.$

$Te_i^+ = \lambda_i^+ e_i^+$

Indeed y has to be of the form $y = \sum_{i=1}^{n+1} a_i e_i^+$

Impose it is $\perp$ to each x_j:
$$0 = \langle y, x_j\rangle = \sum_{i=1}^{n+1} a_i \langle e_i^+, x_j\rangle \quad \forall j = 1,\dots,n$$

n eqs in $n+1$ unknown $\Rightarrow \exists\, y \neq 0$ so that $y \perp V$
Normalize it to get $\|y\| = 1 = \sum |a_i|^2$

$$\leadsto \langle Ty,y\rangle = \sum_{i,j} a_i \overline{a_j} \langle \underset{\lambda_i^+ e_i^+}{\underset{\parallel}{Te_i^+}}, e_j^+\rangle = \sum_i \lambda_i^+ |a_i|^2$$
$$\geq \lambda_{n+1}^+ \underbrace{\sum_i |a_i|^2}_{=1}$$

$$\Rightarrow \sup_{\|x\|=1,\ x\in V^\perp} \langle Tx,x\rangle \geq \lambda_{n+1}^+ \qquad \forall\, V \text{ with } \dim V = n$$

$$\rightsquigarrow \lambda_{n+1}^+ \leq \inf_{V:\dim V=n} \ \sup_{x\in V^\perp,\ \|x\|=1} \langle Tx,x\rangle$$

(=) take $V = (e_1^+,\dots, e_n^+)$ and apply previous corollary to $T|_{V^\perp}$: $\lambda_{n+1}^+ = \max_{x\in V^\perp} \langle Tx,x\rangle$

(ii') Take V with $\dim V = n+1$. Then as before, $\exists\, y \in V$, $\|y\|=1$, $y \in (\operatorname{span}(e_1^+,\dots, e_n^+))^\perp$
(exercise: construct it)

$$\langle Ty,y\rangle \doteq \sum \lambda_i^+ |\langle y,e_i^+\rangle|^2 + \overbrace{\sum \lambda_i^- |\langle y,e_i^-\rangle|^2}^{\leq 0}$$
$$\leq \sum \lambda_i^+ |\langle y,e_i^+\rangle|^2$$
$$= \sum_{i=n+1}^{\infty} \lambda_i^+ |\langle y,e_i^+\rangle|^2 \leq \lambda_{n+1}^+ \|y\|^2 \leq \lambda_{n+1}^+$$

There is at least a $y\in V$ fulfilling the inequality:

$$\rightsquigarrow \min_{x\in V,\ \|x\|=1} \langle Tx,x\rangle \leq \lambda_{n+1}^+ \qquad \forall\, V \text{ with } \dim V = n+1$$

$$\rightsquigarrow \sup_{V:\ \dim V = n+1} \ \min_{x\in V,\ \|x\|=1} \langle Tx,x\rangle \leq \lambda_{n+1}^+$$

To get = and check that sup is achieved choose

$V = \operatorname{span}(e_1, \dots, e_{n+1})$.

Then

$$\min_{x\in V,\ \|x\|=1} \langle Tx,x\rangle \leq \langle Te_{n+1}, e_{n+1}\rangle = \lambda_{n+1}^+$$

$\boxtimes$